Stochastic Differential Equation Models for Systemic Risk

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Introduction

Maintaining financial stability is an important objective of any central bank as failures in the financial system can have a very negative impact on the economy and on the welfare of individuals. Since the great financial crisis, central banks have put increasing focus on understanding and reducing systemic risk in the financial system. We use interacting particle models to study systemic risk in the financial system. In particular, we are interested in studying the effect of different network structures between banks on systemic risk.

Simple model of systemic risk

We start with a simple model of mean-field interacting diffusions to model systemic risk as described in [2]. Let X_t^i denote the log-monetary reserve of agent i at time t. For $i = 1, \dots, N$, the dynamics of each agent are given by:

$$\mathrm{d}X_t^i = \theta_i \left(\bar{X}_t - X_t^i\right) \mathrm{d}t + \sigma_i \, \mathrm{d}W_t^i$$

with initial condition $X_0^i = 0$. W_t^i are independent, standard Brownian motions. The dynamics imply that agent i is attracted to the mean level $\bar{X}_t := \frac{1}{N}\sum_{i=1}^N X_t^i$ at time t. The parameters θ_i represent the strength of mean-field reversion and σ_i the strength of noise for agent i.

For a fixed, exogenous default level $\eta < 0$, a systemic event occurs when the average level of Bank reserves falls below the default level. That is, $\bar{X}_t \le \eta$.

Trajectory of $\bar{X}_t$ and loss distribution for N=10

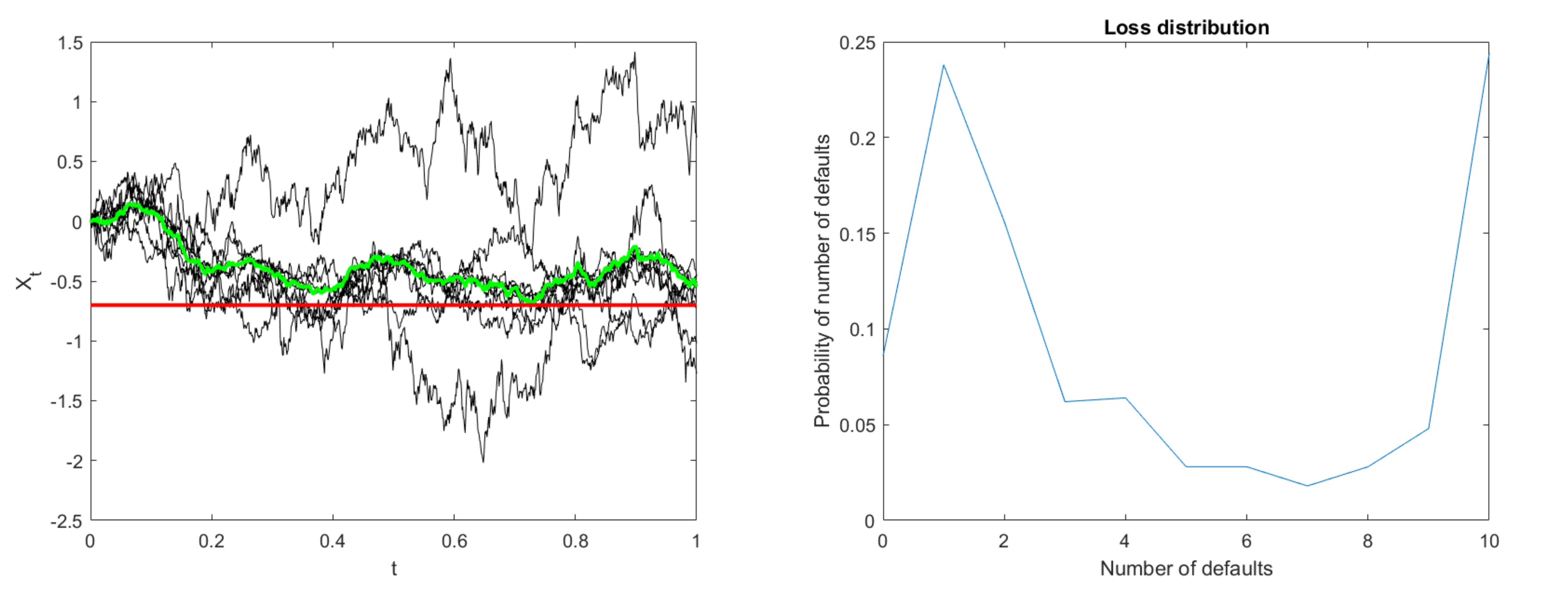

Figure 1. The left graph shows the trajectory of banks' reserves. The red line indicates the default level $\eta = -0.7$. The green line is the trajectory of the empirical mean, $\bar{X}_t$. The right graph shows the estimated 'loss distribution', which is the probability of the number of banks defaulting within the time horizon $[0, 1]$.

Model implications

In this model, the dynamics of all Banks are interconnected. [2] find that:

- Large θ enhances 'flocking' behaviour, where trajectories follow a similar path.
- Larger σ increases the volatility of the system.
- In the cast of different θ_i and σ_i, the system is more stable when the number of intermediate size agents is large enough.

Common noise

We extend the model from [2] to include common noise, in addition to idiosyncratic noise. Economically, common noise is equivalent to aggregate uncertainty in the financial system. The dynamics of each agent are given by:

$$\mathrm{d}X_t^i = \theta_i \left(\bar{X}_t - X_t^i\right) \mathrm{d}t + \sigma_i \, \mathrm{d}\widetilde{W}_t^i,$$

where $\mathrm{d}\widetilde{W}_t^i := \rho \, \mathrm{d}W_t^0 + \sqrt{1-\rho^2}\, \mathrm{d}W_t^i$, with W^i the individual noises and W^0 the common noise and $|\rho| \le 1$.

Implication of common noise

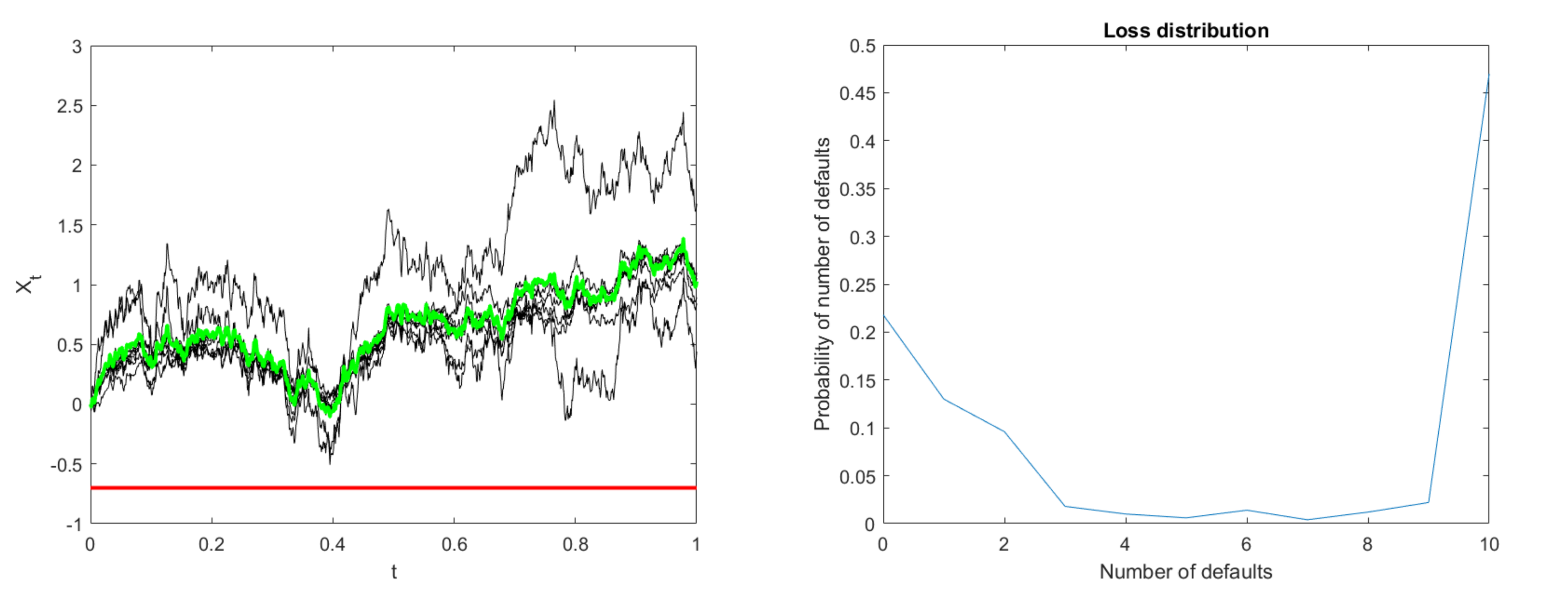

Figure 2. Trajectory of $\bar{X}_t$ and loss distribution with common noise, $\rho = 0.9$.

Common noise enhances 'flocking' behaviour. Including common noise significantly increases the probability of a tail-risk event occurring.

Interbank networks

We extend the model from [2] to incorporate networks on Bank interactions. We express the network as an undirected, weighted adjacency matrix, $\mathbf{A}$, with values $a_{i,j} \in \mathbb{R}^+$ for $i, j \in \{1, \dots, N\}$.

For $i = 1, \dots, N$, the dynamics of each agent are given by:

$$\mathrm{d}X_t^i = \frac{1}{N}\sum_{j=1}^N a_{i,j}\left(X_t^j - X_t^i\right) \mathrm{d}t + \sigma_i \, \mathrm{d}W_t^i$$

Examples of networks

Star-network
In a star-network, some banks are 'central', (fully connected), while other banks are sparsely connected, being connected only to the 'central' banks.

Random network
The graph of a random network is constructed by connecting nodes randomly. The adjacency matrix we consider is symmetric, with ones on the leading diagonal, and other edges being connected with probability p.

Homophilous and heterophilous networks
In a homophilous / heterophilous network, the strength of interaction of two banks is a decreasing / increasing function of the distance in the level of their bank reserves.

Mean-field games

We now add a central bank as a provider of liquidity in the interbank market. [1] develop a game that models interbank borrowing and lending, where each bank controls its rate of borrowing/lending to a central bank. For $i = 1, \dots, N$, the dynamics of each agent are given by:

$$\mathrm{d}X_t^i = \left[\theta\left(\bar{X}_t - X_t^i\right) + \alpha_t^i\right] \mathrm{d}t + \sigma \, \mathrm{d}\widetilde{W}_t^i,$$

where the control α_t^i is chosen to minimise the running quadratic cost f^i and terminal cost g^i:

$$J^i(\alpha) := \mathbb{E}\left[\int_0^T f^i\left(X_t, \alpha_t^i\right) dt + g^i(X_T)\right],$$

with $f^i\left(X_t, \alpha_t^i\right) = \frac{1}{2}\left(\alpha_t^i\right)^2 - q\alpha_t^i\left(\bar{X}_t - X_t^i\right) + \frac{\varepsilon}{2}\left(\bar{X}_t - X_t^i\right)^2$ and $g^i(X_T) = \frac{c}{2}\left(\bar{X}_T - X_T^i\right)^2$

[1] find that introducing the possibility to borrow and lend from a central bank enhances the stability of the financial system.

Approximation of Nash Equilibrium with Deep Learning

The interbank game from [1] falls into the class of N-player *Linear-Quadratic* (LQ) games, which has an analytical expression for the Nash Equilibrium.

We develop a deep neural network-based algorithm to approximate the Nash Equilibrium in order to solve the model extended with network structures.

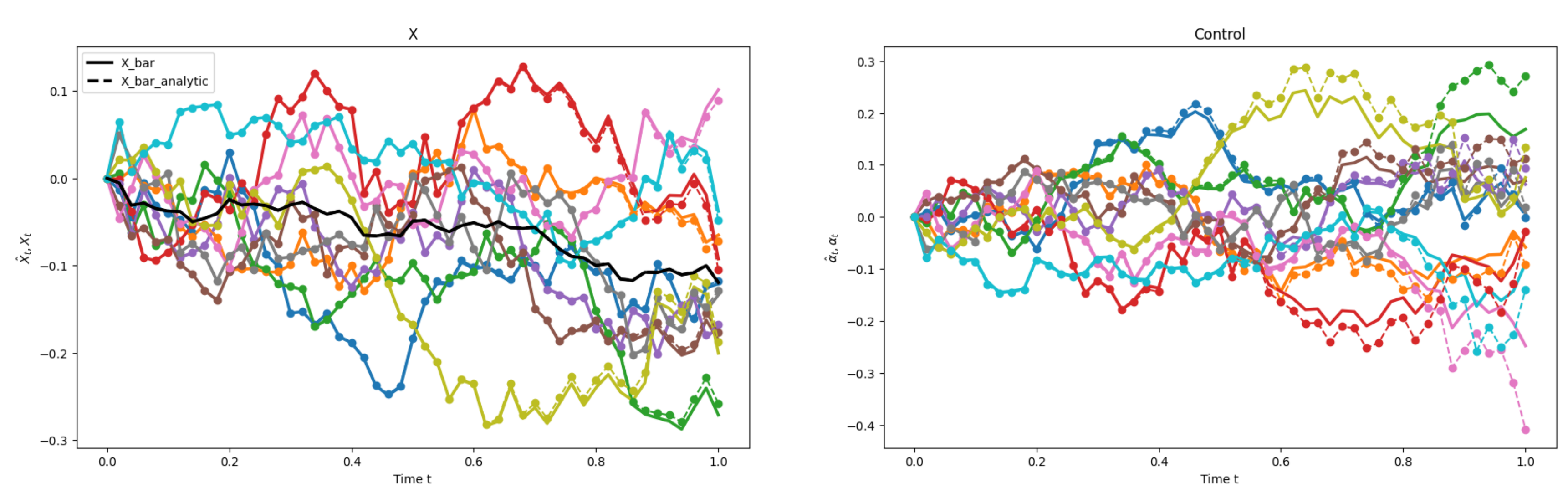

Figure 3. A feedforward neural network with 1 hidden layer approximates the analytical solution of the linear-quadratic game in [1] with $N = 22$. The approximation of the state X_t is very good, while the approximation of the control α_t is also quite good.

Conclusion and policy implications

We analysed interacting particle models of interbank lending to model systemic risk in the financial system. We find that the interbank network structure can have a meaningful impact on the stability of the system. In particular, having many densely-connected intermediate size agents can improve the stability of the financial system. This has policy implications as the central bank may be able to impose regulations to avoid large banks dominating the interbank market.

References

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[2] Fei Fang, Yiwei Sun, and Konstantinos Spiliopoulos. On the effect of heterogeneity on flocking behavior and systemic risk. *Statistics & Risk Modeling*, 34(3-4):141–155, 2017.